Exercise 1 (Explicit finite differences) Type: **Matlab**

We want to solve the linear transport equation $u_t + au_x = 0$ numerically by means of different explicit FD schemes and compare the results.

a) Implement a matlab function

```
function h = num_flux(u_j, u_jp1, a, lambda, method),
```

computing the numerical flux $\texttt{h} = h(u_j, u_{j+1})$ of the method $\texttt{method} \in \{1, 2, 3\}$ with

$$\begin{aligned}
&1)\quad \text{Lax–Friedrichs}: && h(u_j, u_{j+1}) = \frac{1}{2}a(u_{j+1} + u_j) - \frac{1}{2\lambda}(u_{j+1} - u_j),\\
&2)\quad \text{Lax–Wendroff}: && h(u_j, u_{j+1}) = \frac{1}{2}a(u_{j+1} + u_j) - \frac{1}{2}\lambda a^2(u_{j+1} - u_j),\\
&3)\quad \text{Upwind}: && h(u_j, u_{j+1}) = \frac{1}{2}a(u_{j+1} + u_j) - \frac{1}{2}|a|(u_{j+1} - u_j).
\end{aligned}$$

The input parameters $\texttt{u\_j} = u_j$, $\texttt{u\_jp1} = u_{j+1}$ are the approximations from the previous time step, `a` is the coefficient of the transport equation and $\texttt{lambda} = \lambda = \frac{\Delta t}{\Delta x}$.

b) Write a function

```
function u = fdm_explicit(x, t, u_0, u_a, u_b, a, method)
```

that computes the numerical approximation $\texttt{u} = (u_j^l)_{j,l} \approx (u(x_j, t_l))_{j,l}$ by means of the routine of a). The input arguments are the vector $\texttt{x} = (x_j)_{j=0}^{n_x}$ containing the positions of the spatial nodes, the vector $\texttt{t} = (t_l)_{l=0}^{n_t}$ containing the time steps, the vector $\texttt{u\_0} \in \mathbb{R}^{n_x+1}$ specifying the initial conditions at time t_0, as well as the vectors $\texttt{u\_a}, \texttt{u\_b} \in \mathbb{R}^{n_t+1}$ determining the boundary conditions at the boundaries x_0 and x_{n_x}, respectively. Further, `a` is the coefficient of the linear PDE, and `method` determines the FD scheme for the computation of the numerical fluxes. The output argument is a matrix $\texttt{u} \in \mathbb{R}^{(n_x+1)\times(n_t+1)}$ containing the numerical solution $(u_j^l)_{j,l}$ computed according to

$$u_j^{l+1} = u_j^l - \lambda\left(h(u_j^l, u_{j+1}^l) - h(u_{j-1}^l, u_j^l)\right)$$

c) Download the file `transport_test.zip` from the web page, containing the m-files `wave.m`, `fourwaves.m`, `shock.m` as well as the script `transporteq.m`. The latter applies the three methods implemented in a) to the initial conditions in `wave` and plots the results for different values of $\texttt{CFL} = |a|\lambda$.

Execute the script for the initial conditions `wave` and `fourwaves`. Compare the dissipation and dispersion of the three methods with respect to $|a|\lambda$, respectively.

d) Start `transporteq` with the discontinuous initial conditions in `shock` and describe the behaviour of the methods.