Exercise 1 (**Heat equation**) Type: **Matlab Bonus (15 points)**

We want to solve the parabolic PDE

$$u_t = u_{xx}, \qquad u(0,x) = u^0(x), \qquad u(t,0) = u(t,1) = 0, \tag{1}$$

numerically. For this, we first apply a space discretization into $N_x + 1$ equidistant intervals of length $\Delta x = \frac{1}{N_x+1}$, yielding

$$\frac{d}{dt}u_i(t) + \frac{-u_{i-1}(t) + 2u_i(t) - u_{i+1}(t)}{\Delta x^2} = 0, \quad i = 1, \ldots, N_x.$$

The resulting system of ODEs is to be solved by means of the θ-scheme, leading to the following system of linear equations for the computation of the solution vector $\mathbf{u}^{l+1} = (u_1^{l+1}, \ldots, u_{N_x}^{l+1})$ at time t_{l+1}:

$$(\mathrm{Id} + \Delta t \theta A_{\Delta x})\mathbf{u}^{l+1} = (\mathrm{Id} - \Delta t(1-\theta)A_{\Delta x})\mathbf{u}^l. \tag{2}$$

In (2), $A_{\Delta x} \in \mathbb{R}^{N_x \times N_x}$ denotes the Finite Difference matrix from Tutorial 6, Exercise 21.

a) Implement a matlab routine

   `[tvec,xvec,U] = fdm_parabol(INITFUNC,theta,Tint,Xint,Nt,Nx),`

   which solves the PDE (1) by means of the discretization (2) for `theta` $= \theta$. The input arguments are the space interval `Xint` $= [x_0, x_{N_x+1}]$ and the time interval `Tint` $= [t_0, t_{N_t}]$ which are partitioned into N_x inner nodes and N_t time steps, respectively. `INITFUNC` contains the function handle for the initial conditions.
   The output arguments are the time step vector `tvec` of length $N_t + 1$, the space step vector `xvec` with $N_x + 2$ entries including the boundary nodes, as well as the discrete solution `U` $= (u_i^l)_{il} \in \mathbb{R}^{(N_x+2)\times(N_t+1)}$.

b) Download the folder `parabol_test.zip` from the web page and start the script `heat_test.m` with the parameters $\theta = 0.9$ and $N_t = 20$. Then, perform the same calculation with $\theta = 0.1$ as well as 20 and 100 time steps, respectively. Interpret your results.